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Evolution of Governing Mass and Momentum Balances Following an Abrupt Pressure Impact in a Porous Medium

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Abstract. A mathematical model is developed of an abrupt pressure impact applied to a compressible fluid flowing through a porous medium domain. Nondimensional forms of the macroscopic fluid mass and momentum balance equations yield two new scalar numbers relating storage change to pressure rise. A sequence of four reduced forms of mass and momentum balance equations are shown to be associated with a sequence of four time periods following the onset of a pressure change. At the very first time period, pressure is proven to be distributed uniformly within the affected domain. During the second time interval, the momentum balance equation conforms to a wave form. The behavior during the third time period is governed by the averaged Navier–Stokes equation. After a long time, the fourth period is dominated by a momentum balance similar to Brinkman's equation which may convert to Darcy's equation when friction at the solid-fluid interface dominates.

Key words. Compressible fluid, porous media, abrupt pressure change, time and spatial averaging, mass and momentum balance equations, nondimensional forms.

1. Introduction

In a private communication, Professor Arie Issar* suggested that due to the fact that in a confined aquifer, pressure propagates much faster than fluid mass, the rise in water levels should be described by a wave equation without resorting to Darcy's law. This comment initiated the research described in this paper.

The momentum balance equation, for example, in the form

$$\frac{\partial(\rho v_k)}{\partial t} = -\frac{\partial}{\partial x_i}(\rho v_i v_k) + \frac{\partial \tau_{ik}}{\partial x_i} - \frac{\partial p}{\partial x_i}\delta_{ik} + \rho F_k, \tag{1}$$

serves as a starting point for analyzing the motion of a single phase fluid and spatial and temporal changes that may take place in the flow regime, due to various excitations. In (1), v_i denotes the ith component of the mass weighted

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velocity vector of the (possibly inhomogeneous) fluid, τ_{ik} denotes the ikth component of the fluid's shear stress tensor, δ_{ik} denotes the Kronecker delta, ρ denotes fluid's density, and F_k denotes the kth component of the body force vector acting on the fluid, per unit mass. Note that in (1), $\sigma_{ik} = \tau_{ik} - p\delta_{ik}$ is the stress tensor, with σ_{ij} expressing the diffusive flux of momentum.

The constitutive equation for a compressible Newtonian fluid is expressed by

$$\tau_{ik} = \mu\left(\frac{\partial \nu_i}{\partial x_k} + \frac{\partial \nu_k}{\partial x_i}\right) + \lambda'' \frac{\partial \nu_j}{\partial x_j}\,\delta_{ik}, \tag{2}$$

where μ denotes the fluid's viscosity and $(\lambda'' + \frac{2}{3}\mu)$ is called the second viscosity coefficient. By substituting (2) into (1), we obtain

$$\frac{\partial(\rho\nu_k)}{\partial t} + \frac{\partial}{\partial x_i}(\rho\nu_i\nu_k) = \rho F_k - \frac{\partial p}{\partial x_i}\,\delta_{ik} + \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial \nu_i}{\partial x_k} + \frac{\partial \nu_k}{\partial x_i}\right) + \lambda'' \frac{\partial \nu_j}{\partial x_j}\,\delta_{ik}\right]. \tag{3}$$

Note that often in the literature, the momentum balance equation (3) is presented in the form

$$\rho\left(\frac{\partial \nu_k}{\partial t} + \nu_i \frac{\partial \nu_k}{\partial x_i}\right) = \rho F_k - \frac{\partial p}{\partial x_i}\,\delta_{ik} + \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial \nu_i}{\partial x_k} + \frac{\partial \nu_k}{\partial x_i}\right) + \lambda'' \frac{\partial \nu_j}{\partial x_j}\,\delta_{ik}\right], \tag{4}$$

referred to as the Navier–Stokes equation. This form is obtained by subtracting from the momentum balance equation (3), the mass balance equation, without source terms, given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\nu_i)}{\partial x_i} = 0, \tag{5}$$

multiplied by the fluid's velocity, ν_k.

Each term in (1) represents a rate of momentum added, per unit fluid volume. For example, the first term on the right-hand side, describes the rate of momentum gained by the fluid's advection, while the sum of the second and third terms represents the momentum gained by diffusive momentum transfer.

At this stage, it is common to analyze the order of magnitude of the various terms appearing in (4), relative to each other. The objective is to simplify the equation whenever certain terms can be shown to be nondominant, i.e. much smaller than other terms and, therefore, can be neglected. For example, in this way, an equation is obtained that describes the behavior of a fluid in a boundary layer, close to a solid surface, or farther away from that surface. Another example is the equation that describes the propagation of pressure waves.

This paper will analyze the evolution in time, of the flow regime in a porous medium domain, in response to an abrupt pressure change at a point on the boundary, or an abrupt change in the strength of a point sink, or source. An explosion in a well penetrating a porous medium domain, may serve as an example. We shall achieve this goal by analyzing the changes in time that take place in the order of magnitude of the various terms that appear in the (macroscopic) momentum balance equation that describes the motion equations in a porous medium.

As a starting point, we shall use the macroscopic momentum balance equation developed by Bear and Bachmat (1986), modifying it to the form

$$n\rho \frac{\partial v_i}{\partial t} - 2n\rho v_j w_{ij} + n\rho \frac{\partial}{\partial x_i}\left(\frac{v^2}{2}\right) + n\frac{\partial p}{\partial x_j} T_{ij} + n\rho g \frac{\partial Z}{\partial x_j} T_{ij} +$$

$$-\mu\left(\frac{\partial^2 q_{ri}}{\partial x_j\, \partial x_j} + \frac{\partial^2 q_{rj}}{\partial x_i\, \partial x_j}\right) - \lambda'' \frac{\partial^2 q_{ri}}{\partial x_j\, \partial x_j} + \frac{\mu C_f}{\Delta_f^2} \alpha'_{ij} q_{rj} = 0. \tag{6}$$

In this equation, obtained by averaging (4) over a Representative Elementary Volume (REV), n denotes the porosity, defined as the ratio of the pore volume to the REV volume, T_{ij} denotes the ijth component of the tortuosity tensor, g denotes the gravity acceleration, Z denotes an elevation (i.e. gravity is introduced as the body force, F_i), q_r denotes the specific flux relative to the solid phase, C_f denotes a shape factor, Δ_f denotes the hydraulic radius of the pore space, and w_{ij} denotes the vorticity, defined as

$$w_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right). \tag{7}$$

The coefficient α'_{ij} (typical component of the tensorial coefficient) is defined by

$$\alpha'_{ij} = \delta_{ij} + \left(1 + \frac{\lambda''}{\mu}\right) \overline{\xi_i \xi_j}^{sf}, \tag{8}$$

where $\overline{\xi_i \xi_j}^{sf}$ denotes the average, over the solid-fluid interface, of the product of the directional cosines of the outer normal unit vector ξ_i. Note that for an incompressible fluid, $\alpha'_{ij} = \delta_{ij} + \overline{\xi_i \xi_j}^{sf}$.

Let us briefly summarize how Bear and Bachmat (1986) derive Equation (6). They start from the momentum balance equation (4), written at the microscopic level, i.e., at a point inside a fluid phase that occupies the void space. In order to transform (4) into the macroscopic, or averaged, momentum balance equation (6), they make use of the following averaging rules:

Average of an intensive variable, e^α, of an α phase with a REV is given by

$$\bar{e}^\alpha = \frac{1}{U_\alpha} \int_{U_0} e^\alpha \, \mathrm{d}U_0, \tag{9.1}$$

where U_0 is the volume of the REV and U_α denotes the volume of the α phase within U_0.

Average of a time derivative

$$n\frac{\partial \bar{e}^\alpha}{\partial t} = \frac{\partial n \bar{e}^\alpha}{\partial t} + \frac{1}{U_0} \int_{S_{\alpha s}} e\mathbf{u} \cdot \boldsymbol{\xi} \, \mathrm{d}S, \tag{9.2}$$

where $S_{\alpha s}$ denotes the interface between the solid (S), and the α-phase within the REV and $\mathbf{u}$ is the velocity of the (possibly moving) $\alpha - s$ surface.

Average of a spatial derivative

$$\theta_\alpha \frac{\partial \bar{e}^\alpha}{\partial x_i} = \frac{\partial \theta_\alpha \bar{e}^\alpha}{\partial x_i} + \frac{1}{U_0} \int_{S_{\alpha s}} e_\alpha \xi_i \, \mathrm{d}S, \tag{9.3}$$

where θ_α denotes the volumetric fraction of α within REV. When dealing with a fluid phase ($\alpha \equiv f$) that occupies the entire void space, θ_α represents the porosity n.

Modified rule for averaging a spatial derivative

$$\frac{\partial \bar{e}^\alpha}{\partial x_i} = \frac{\partial \bar{e}^\alpha}{\partial x_j} T_{\alpha ij} + \frac{1}{nU_0} \int_{S_{\alpha s}} \mathring{x}_i \frac{\partial e}{\partial x_j} \xi_j \, \mathrm{d}S, \tag{9.4}$$

where $\mathring{x}_i$ denotes the ith coordinate of a point in the REV relative to the REV's centroid. The modified rule is valid for a case where $\nabla^2 e = 0$ in the fluid. Bear and Bachmat (1986) assume that this assumption can be justified for fluid pressure. The tensorial tortuosity coefficient, T_{ij}, appearing in (6) and (9.4), is defined by

$$T_{\alpha ij} = \frac{1}{nU_0} \int_{S_{\alpha\alpha}} \mathring{x}_i \xi_j \, \mathrm{d}S, \tag{10}$$

where $S_{\alpha\alpha}$ denotes the α–α part of the boundary of the REV. This macroscope coefficient reflects the microscopic configuration of the solid-fluid interface. The main assumptions made by Bear and Bachmat (1986) in order to obtain (6) are

- the fluid is Newtonian and compressible;
- close to the solid–fluid interface, the components of the resistance force normal to this surface, $\partial \tau_{ij}/\partial x_j$, are much smaller than the tangential ones, and, therefore, the pressure distribution there is hydrostatic;
- the fluid adheres to the solid and the component of the velocity gradient normal to the fluid–solid interface, averaged over this entire interface, is approximated by

$$\frac{1}{U_0} \int_{S_{fs}} \frac{\partial v_j}{\partial s_\xi} \mathrm{d}S \cong - C_f n \frac{\bar{v}_j^f - \bar{v}_{s_j}^s}{\Delta_f^2}. \tag{11}$$

In a similar way, a macroscopic fluid mass balance equation is obtained by averaging Equation (5), neglecting the diffusive and dispersive fluxes of the total fluid mass

$$\frac{\partial (n\rho)}{\partial t} + \frac{\partial (n\rho v_i)}{\partial x_i} = 0. \tag{12}$$

By assuming that the porous medium is initially homogeneous with respect to porosity, but the latter is a function of pressure, the mass balance equation (12) can be written in the form

$$S_0 \frac{\partial p}{\partial t} + n \frac{\partial v_j}{\partial x_j} + n\beta v_j \frac{\partial p}{\partial x_j} = 0, \tag{13}$$

where we assume

$$\left| \rho \frac{\partial n}{\partial t} \right| \gg \left| \rho v_i \frac{\partial n}{\partial x_i} \right|$$

and (14)

$$S_0 = n\beta + (1-n)\alpha$$

denotes the specific storativity of the porous medium,

$$\beta = \frac{1}{\rho} \frac{\partial \rho}{\partial p} \tag{15}$$

denotes the fluid's compressibility, and

$$\alpha = \frac{1}{1-n} \frac{\partial n}{\partial p} \tag{16}$$

Denotes the coefficient of compressibility of the porous medium as a whole (based on the assumption that the effect of shear in the fluid is negligible and that deformation of the solid matrix is mainly in the form of vertical compaction).

Note that actually, porosity is a function of effective stress. However, it may be expressed as a function of fluid pressure only, if we assume, in addition to the above assumptions, that no changes take place in the total stress.

By combining (13) multiplied by fluid's velocity and (6), the macroscopic momentum balance (3) can be rewritten in the form

$$n\rho \frac{\partial v_i}{\partial t} + S_0 \rho v_i \frac{\partial p}{\partial t} + 2\rho n \left[\frac{\partial}{\partial x_i} \left(\frac{v^2}{2} \right) - 2 v_j w_{ij} \right] + n(T_{ij} + \rho\beta v_i v_j) \frac{\partial p}{\partial x_j} +$$

$$+ n\rho g \frac{\partial Z}{\partial x_j} T_{ij} - \mu \left(\frac{\partial^2 q_{ri}}{\partial x_j \, \partial x_j} + \frac{\partial^2 q_{rj}}{\partial x_i \, \partial x_j} \right) - \lambda'' \frac{\partial^2 q_{ri}}{\partial x_i \, \partial x_j} + \frac{\mu C_f}{\Delta_f^2} \alpha'_{ij} q_{rj} = 0. \tag{17}$$

Our objective in what follows, is to analyze the order of magnitude of the various terms appearing in [17].

2. Nondimensional Forms of the Mass and Momentum Balance Equations

To achieve the above goal, our next step is to rewrite (13) and (17) in non-dimensional forms.

The goal of a nondimensional representation of an equation is to help in identifying nondominant terms so that they may be neglected. This will lead to an approximate new governing equation. Eliminating terms is based on a comparison between the scalar factors that multiply nondimensional terms involving the dependent variables, and their derivatives which are of *order one* (denoted as $O(1)$). The selection of the *scales* that are used in order to nondimensionalize the given (dimensional) equation, is such as to ensure that the latter terms are indeed of order one. For example, a length scale, L_c^p is selected to represent spatial changes in p. Similarly, L_c^ν represents a length characterizing the spatial changes in velocity. In a similar way, we select Δt_c^p and Δt_c^ν as time intervals characterizing the temporal changes in pressure and in velocity, respectively. Note that because we have neglected ∇n, $L_c^\nu \equiv L_c^q$.

For example, for an intensive quantity, e, we select a characteristic change, e_c, and a characteristic length, L_c^e, such that

$$\frac{\partial e}{\partial x_i} = \frac{e_c}{L_c^e}\left[\frac{\partial e}{\partial x_i}\right]^*, \tag{18.1}$$

where the asterisk denotes a nondimensional value of the argument. Similarly, for a time derivative

$$\frac{\partial e}{\partial t} = \frac{e_c}{\Delta t_c^e}\left[\frac{\partial e}{\partial t}\right]^*. \tag{18.2}$$

To simplify our symbols, we use e_c and L_c (not Δe_c and Δx_c) to designate a characteristic increment in the intensive quantity and in the spatial distance, respectively.

Let us now express the product of the fluid's velocity and its mass balance equation given by (13) in a nondimensional form. We obtain

$$(\nu_c\nu_i^*)\left\{\frac{S_0 p_c}{\Delta t_c^p}\left[\frac{\partial p}{\partial t}\right]^* + \frac{n_c\nu_c}{L_c^\nu}\left[\frac{\partial \nu_j}{\partial x_j}\right]^* + \frac{p_c n_c \beta \nu_c}{L_c^p}\left[\nu_j\frac{\partial p}{\partial x_j}\right]^*\right\} = 0. \tag{19}$$

Equation (19) can be rewritten to the form

$$\frac{\nu_c S_0 \rho_c p_c}{\Delta t_c^p}\left[\rho\nu_i\frac{\partial p}{\partial t}\right]^* + \frac{n_c\rho_c\nu_c^2}{L_c^\nu}\left[n\rho\left(\frac{\partial}{\partial x_i}\left(\frac{\nu^2}{2}\right) - 2\nu_j w_{ij}\right)\right]^* +$$
$$+\frac{n_c\nu_c^2\rho_c\beta p_c}{L_c^p}\left[n\rho\nu_i\nu_j\frac{\partial p}{\partial x_j}\right]^* = 0. \tag{20}$$

Similarly, by virtue of (17) and (18), we obtain a nondimensional expression for the fluid's momentum balance, in the form

$$\frac{n_c\rho_c\nu_c}{\Delta t_c^\nu}\left[n\rho\frac{\partial \nu_i}{\partial t}\right]^* + \frac{\nu_c S_0\rho_c p_c}{\Delta t_c^p}\left[\rho\nu_i\frac{\partial p}{\partial t}\right]^* +$$
$$+\frac{2n_c\rho_c\nu_c^2}{L_c^\nu}\left[n\rho\left(\frac{\partial}{\partial x_i}\left(\frac{\nu^2}{2}\right) - 2\nu_j w_{ij}\right)\right]^* + \frac{n_c\rho_c\beta\nu_c^2 p_c}{L_c^p}\left[n\rho\nu_i\nu_j\frac{\partial p}{\partial x_j}\right]^* +$$
$$+\frac{n_c p_c}{L^p}\left[n\frac{\partial p}{\partial x_i}\right]^* T_{ij} + n_c\rho_c g\left[n\rho\frac{\partial Z}{\partial x_i}\right]^* T_{ij} - \frac{\mu_c q_{rc}}{(L_c^\nu)^2}\left[\mu\left(\frac{\partial^2 q_{ri}}{\partial x_j\,\partial x_i} + \frac{\partial^2 q_{rj}}{\partial x_i\,\partial x_j}\right)\right]^* +$$

$$-\frac{\lambda''_c q_{rc}}{(L_c^\nu)^2}\left[\lambda''\frac{\partial^2 q_{ri}}{\partial x_j\,\partial x_j}\right]^* + \frac{\mu_c C_f}{\Delta_f^2} q_{rc}\alpha'_{ij}[\mu q_{rj}]^* = 0, \tag{21}$$

where q_{rc} denotes the characteristic specific flux, defined by

$$q_{rc} = n_c v_c. \tag{22.1}$$

Furthermore, we introduce the following dimensionless numbers:

Struhal number St^e, given by

$$\mathrm{St}^e = \frac{L_c^e}{\Delta t_c^e v_c}. \tag{22.2}$$

Note that we are making use of the well known Struhal number, but emphasize that we have to associate a different Struhal number for every intensive quantity, e.

Darcy number, Da, is given by

$$\mathrm{Da} = \frac{k_c/n_c}{(L_c^\nu)^2}, \tag{22.3}$$

where k_c denotes a characteristic permeability defined by

$$k_c = \frac{n_c \Delta_f^2}{C_f}. \tag{22.4}$$

Reynolds number, Re, is given by

$$\mathrm{Re} = \frac{v_c\sqrt{k_c/n_c}}{\upsilon_c} = \frac{v_c L_c^\nu\,\mathrm{Da}^{-1/2}}{\upsilon_c}, \tag{22.5}$$

where υ_c denotes the characteristic kinematic viscosity, defined by

$$\upsilon_c = \frac{\mu_c}{\rho_c}. \tag{22.6}$$

Note that the Re number here is also somewhat different from that commonly employed in porous media studies.

Euler number, Eu, is given by

$$\mathrm{Eu} = \frac{p_c}{\rho_c v_c^2}. \tag{22.7}$$

Richardson number, Ri, is given by

$$\mathrm{Ri} = \mathrm{Fr}^{-2} = \frac{g L_c^\nu}{v_c^2}, \tag{22.8}$$

where Fr denotes the *Froude number*.

Our next objective is to estimate the magnitude of terms in the fluid's mass and momentum balance equations, relative to each other. More specifically, our goal in this paper is to conduct such an evaluation for a flow regime produced by an abrupt pressure change, and changes in these regimes in the course of time. To do so, we divide all terms in the mass and momentum balance equations by the scalar factor multiplying the nondimensional rate of pressure change. In view of (22), Equations (20) and (21) are rewritten, respectively, in the form

$$\left[\rho v_i \frac{\partial p}{\partial t}\right]^* + \overset{A_2}{U_p^{-1}}(\mathrm{St}^\nu)^{-1}\frac{\Delta t_c^p}{\Delta t_c^\nu}\left[n\rho\left(\frac{\partial}{\partial x_i}\left(\frac{\nu^2}{2}\right) - 2\nu_j w_{ij}\right)\right]^* + \overset{A_4}{+ U_\beta^{-1}}(\mathrm{St}^\nu)^{-1}\frac{L_c^\nu}{L_c^p}\frac{\Delta t_c^p}{\Delta t_c^\nu}\left[n\rho\nu_i\nu_j \frac{\partial p}{\partial x_j}\right]^* = 0, \tag{23}$$

and

$$U_p^{-1}\frac{\Delta t_c^p}{\Delta t_c}\left[n\rho\frac{\partial \nu_i}{\partial t}\right]^* + \left[\nu_i\rho\frac{\partial p}{\partial t}\right]^* + 2U_p^{-1}(\mathrm{St}^\nu)^{-1}\frac{\Delta t_c^p}{\Delta t_c^\nu}\left[n\rho\left(\frac{\partial}{\partial x_i}\left(\frac{\nu^2}{2}\right) - 2\nu_j w_{ij}\right)\right]^* +$$
$$+ U_\beta^{-1}(\mathrm{St}^\nu)^{-1}\frac{L_c^\nu}{L_c^p}\frac{\Delta t_c^p}{\Delta t_c^\nu}\left[n\rho\nu_i\nu_j\frac{\partial p}{\partial x_j}\right]^* + U_p^{-1}\mathrm{Eu}(\mathrm{St}^\nu)^{-1}\frac{L_c^\nu}{L_c^p}\frac{\Delta t_c^p}{\Delta t_c^\nu}\left[n\frac{\partial p}{\partial x_j}\right]^* T_{ij} +$$
$$+ U_p^{-1}R_I(\mathrm{St}^\nu)^{-1}\frac{\Delta t_c^p}{\Delta t_c^\nu}\left[n\rho\frac{\partial Z}{\partial x_j}\right]^* T_{ij} +$$
$$+ U_p^{-1}\mathrm{Re}^{-1}\mathrm{Da}^{1/2}(\mathrm{St}^\nu)^{-1}\frac{\Delta t_c^p}{\Delta t_c^\nu}\left\{\mu\left[\frac{\partial^2 q_{ri}}{\partial x_j\,\partial x_j} + \frac{\partial^2 q_{rj}}{\partial x_i\,\partial x_j}\right]^* + \left(\frac{\lambda_c''}{\mu_c}\right)\left[\lambda''\frac{\partial^2 q_{ri}}{\partial x_j\,\partial x_j}\right]^*\right\} +$$
$$+ U_p^{-1}\mathrm{Re}^{-1}\mathrm{Da}^{-1/2}(\mathrm{St}^\nu)^{-1}\frac{\Delta t_c^p}{\Delta t_c^\nu}[\mu q_{rj}]^*\alpha'_{ij} = 0. \tag{24}$$

Note the two new scalar numbers U_p and U_β appearing in (23) and (24). The first, U_p, defined as

$$U_p = \frac{S_0 p_c}{n_c}, \tag{25}$$

gives the volume of fluid added to storage, per unit volume of fluid (fully occupying the void space) as pressure increases (due to the compressibility of the fluid and the porous medium as a whole). The second one,

$$U_\beta = \frac{S_0/\beta}{n_c} = \frac{S_0 \Big/ \frac{1}{\rho}\frac{\partial \rho}{\partial p}}{n_c} = \frac{S_0\, P_c}{n_c\left(\frac{\partial \rho^*}{\partial p^*}\right)} \tag{26}$$

represents the storativity of the entire system, relative to that contributed by the fluid alone.

Next, we evaluate the changes that evolve (in time) in the relative magnitude of the terms appearing in the mass and momentum balance equations.

3. Approximate Forms of Mass and Momentum Balance Equations

To simplify the discussion, without affecting the general ideas we wish to present, let us consider a fluid that is initially at rest and is then subject to an abrupt change in pressure at a point within, or on the boundary of a porous domain. In such a case, for some time after the pressure change has been applied, we may assume that

$$\Delta t_c^{\nu} = \Delta t_c^{p}, \tag{27.1}$$

i.e. the time intervals characterizing pressure and velocity changes, are identical. The characteristic velocity of fluid's motion, ν_c and the characteristic speed of pressure propagation ν_c^p, may be expressed by

$$\nu_c = \frac{L_c^{\nu}}{\Delta t_c^{\nu}}; \quad (\mathrm{St}^{\nu} \equiv 1), \tag{27.2}$$

$$\nu_c^p = \frac{L_c^p}{\Delta t_c^p}. \tag{27.3}$$

Note that only during the short time interval following the initiation of motion from rest, the characteristic velocities are identical to the ratio between the characteristic length and the characteristic time and, hence, the Struhal number with reference to velocity is a unit.

In view of (23), (24) and (27), we define a number of factors

$$A_1 \equiv U_p^{-1} \frac{\Delta t_c^p}{\Delta t_c^{\nu}} = U_p^{-1}, \tag{28.1}$$

$$A_2 \equiv 2U_p^{-1}(\mathrm{St}^{\nu})^{-1} \frac{\Delta t_c^p}{\Delta t_c^{\nu}} = 2U_p^{-1}, \quad = \frac{2}{U_p} \tag{28.2}$$

$\frac{V_c}{U_p} \sim \frac{\ }{V_c^p}$

$$A_3 \equiv U_p^{-1} E_u (\mathrm{St}^{\nu})^{-1} \frac{L_c^{\nu} \Delta t_c^p}{L_c^p \Delta t_c^{\nu}} = \frac{n_c}{S_0 \rho_c \nu_c \nu_c^p}, \tag{28.3}$$

$$A_4 \equiv U_\beta^{-1} (\mathrm{St}^{\nu})^{-1} \frac{L_c^{\nu} \Delta t_c^p}{L_c^p \Delta t_c^{\nu}} = U_\beta^{-1} \frac{\nu_c}{\nu_c^p}, \quad \frac{1}{U_\beta} \quad \frac{V_c}{V_c^p} \tag{28.4}$$

$$A_5 \equiv U_p^{-1} \,\mathrm{Ri} (\mathrm{St}^{\nu})^{-1} \frac{\Delta t_c^p}{\Delta t_c^{\nu}} = U_p^{-1} \frac{g L_c^p}{\nu_c \nu_c^p}, \tag{28.5}$$

$$A_6 \equiv U_p^{-1} \,\mathrm{Re}^{-1} \,\mathrm{Da}^{1/2} (\mathrm{St}^{\nu})^{-1} \frac{\Delta t_c^p}{\Delta t_c^{\nu}} = U_p^{-1} \frac{v_c \Delta t_c^p}{(L_c^{\nu})^2}, \tag{28.6}$$

$$A_7 \equiv U_p^{-1} \alpha'_{ij} \mathrm{Re}^{-1} \mathrm{Da}^{-1/2} (\mathrm{St}^{\nu})^{-1} \frac{\Delta t_c^p}{\Delta t_c^{\nu}} = U_p^{-1} \alpha'_{ij} \frac{v_c}{(k_c/n_c)} \Delta t_c^p. \tag{28.7}$$

We now estimate different forms, evolving as time proceeds, of the mass and momentum balance equations. The various forms are constructed by considering the relative order of magnitude of the A_m values ($m = 1, 2, \ldots, 7$), defined by (28).

3.1. PERIOD OF UNIFORM PRESSURE DISTRIBUTION WITHIN THE DOMAIN

We confine our discussion to the period immediately following the pressure impulse. In such a case

$$L_c^v \cong L_c^p \to 0 \;@\; \Delta t^p = 0^+. \tag{29}$$

Note that $\Delta t^p = 0^+$ may be regarded as the duration in the assumed abrupt rise in pressure.

Accordingly, by virtue of (28) and (29), we obtain

$$A_3 \cong 0(1), \qquad A_4 \cong 0(1) \tag{30}$$

while all other A's are much smaller than A_3 and A_4.

With relation (30) in mind, we rewrite (13) and (17) as the dominating balance equations,

$$S_0\rho \frac{\partial p}{\partial t} + n\rho\beta\nu_i \frac{\partial p}{\partial x_i} = 0, \tag{31}$$

for the mass balance, and

$$S_0\rho\nu_i \frac{\partial p}{\partial t} + n\rho\beta\nu_i\nu_j \frac{\partial p}{\partial x_j} + n \frac{\partial p}{\partial x_j} T_{ij} = 0, \tag{32}$$

for the momentum balance resulting from (3).

However, subtracting (31) multiplied by the fluid's velocity from (32), yields a momentum balance equation (related to (6)) that takes the form

$$n \frac{\partial p}{\partial x_j} T_{ij} = 0. \tag{33}$$

The solution of (31) and (33) is one of uniform pressure distribution

$$p|_{\Delta t = 0^+} = \text{const.} \tag{34.1}$$

Note that the physical significance of (34.1) is that at the very first instant of applying a pressure impulse, the compressible fluid acts as an incompressible one. The pressure impulse propagates without attenuation over a distance obtained from the A_3 factor in (30).

$$L_c^p = \frac{n_c}{p_c S_0 \nu_c} \Delta t_c^p. \tag{34.2}$$

3.2. PERIOD OF PRESSURE WAVE PROPAGATION

We now proceed further in time. The characteristic length, L_c^p, associated with pressure changes, is greater than the one, L_c^v, associated with changes in the fluid velocity, i.e.

$$L_c^p > L_c^v. \tag{35.1}$$

The considered period is still sufficiently small, so as to make $\mathrm{St}^v = 1$. Hence

$$v_c^p > v_c, \tag{35.2}$$

and in view of (28) and (35), we obtain

$$A_1 \cong O(1), \qquad A_2 \cong O(1), \qquad A_3 \cong O(1), \qquad A_4 \cong O(1), \qquad A_5 \cong O(1), \tag{36.1}$$

while

$$A_6, A_7 \ll O(1). \tag{36.2}$$

In view of (23) and (36), the mass balance equation (13) remains unchanged. By multiplying the mass balance (13), by the fluid's velocity, we obtain

$$S_0 \rho v_i \frac{\partial p}{\partial t} + n\rho\left[\frac{\partial}{\partial x_i}\left(\frac{v^2}{2}\right) - 2 v_i w_{ij}\right] + n\rho\beta v_i v_j \frac{\partial p}{\partial x_j} = 0. \tag{37}$$

By virtue of (24) and (36), we obtain a momentum balance equation in the form

$$n\rho \frac{\partial v_i}{\partial t} + 2n\rho\left[\frac{\partial}{\partial x_i}\left(\frac{v^2}{2}\right) - 2 v_i w_{ij}\right] + S_0 \rho v_i \frac{\partial p}{\partial t} + n\rho\beta v_i v_j \frac{\partial p}{\partial x_j} + $$
$$+ n\left[\frac{\partial p}{\partial x_j} + \rho g \frac{\partial Z}{\partial x_j}\right] T_{ij} = 0. \tag{38}$$

Equation (38) is a consequence of (3). However, if we subtract (37) from (38), we obtain a momentum balance originating from (6), in the form

$$\rho \frac{\partial v_i}{\partial t} + \rho\left[\frac{\partial}{\partial x_i}\left(\frac{v^2}{2}\right) - 2 v_i w_{ij}\right] + \left[\frac{\partial p}{\partial x_j} + \rho g \frac{\partial Z}{\partial x_j}\right] T_{ij} = 0. \tag{39}$$

If we now assume translatory flow only, without vorticity, i.e.

$$w_{ij} = 0, \tag{40}$$

we obtain a momentum balance of the form

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_i}\left(\frac{v^2}{2}\right) + \left[\frac{1}{\rho}\frac{\partial p}{\partial x_j} + g\frac{\partial Z}{\partial x_j}\right] T_{ij} = 0. \tag{41}$$

Realizing that velocity represents a time derivative of spatial displacement, (41) then describes a wave equation.

Note that the momentum balance (41) can also be obtained when the fluid flows at a high Reynolds number, due to high velocities (e.g., Kivity and Collins, 1973; Barta *et al.*, 1982). However, in our case, the fluid's velocity is low, but the gradient of that velocity is steep, hence leading to the wave form expressed in (41).

The question as to the extent of time over which (39) or (41) are valid, may be addressed by expressions (27.3) and factor A_3 (or A_5) in (36.1).

$$0 < \Delta t_c^p \leq U_p \frac{\rho_c \nu_c}{p_c} L_c^p. \tag{42}$$

We now introduce Hubbert's potential, Φ, defined by

$$\Phi = \int_{p_0}^{p} \frac{\mathrm{d}p}{\rho(p)g} + Z, \tag{43}$$

where p_0 denotes a reference pressure. In view of (43) the spatial and time derivatives of Φ, will be

$$\frac{\partial \Phi}{\partial x_j} = \frac{1}{\rho g} \frac{\partial p}{\partial x_j} + \frac{\partial Z}{\partial x_j}, \tag{44}$$

$$\frac{\partial \Phi}{\partial t} = \frac{1}{\rho g} \frac{\partial p}{\partial t}. \tag{45}$$

Upon substituting (44), (45)) into (13) and (44) into (41), we obtain mass and monentum balance equations, respectively, in the form

$$S_0 \rho g \frac{\partial \Phi}{\partial t} + \rho g n \beta \nu_j \frac{\partial}{\partial x_j} (\Phi - Z) + n \frac{\partial \nu_j}{\partial x_j} = 0, \tag{46}$$

$$\frac{\partial \nu_i}{\partial t} + \frac{\partial}{\partial x_i} \left(\frac{\nu^2}{2} \right) + g \frac{\partial \Phi}{\partial x_j} T_{ij} = 0. \tag{47}$$

We now define a material derivative defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \nu_j \frac{\partial}{\partial x_j}. \tag{48}$$

The momentum balance (39), including the vorticity term may then be written in the form

$$\frac{D\nu_i}{Dt} + g \frac{\partial \Phi}{\partial x_j} T_{ij} = 0. \tag{49}$$

Note the difference in approaches represented by (39) and (49). Equation (39) is written from an Eulerian point of view. i.e., velocity is measured from a fixed spatial reference as fluid passes this location. However, (49) is written from a Lagrangian point of view, i.e. velocity is measured along a pathline, say, of a fluid particle that moves along this characteristic line. The pathline equation is given by

$$\frac{Dx_i}{Dt} = v_i. \tag{50}$$

We now define a time step, Δt. which is the difference between the forward time level, t^{k+1}, and the backward one, t^k.

$$\Delta t = t^{k+1} - t^k. \tag{51}$$

We approximate (50) by

$$x_i^{k+1} = x_i^k + \int_t^{t+\Delta t} v_i Dt, \tag{52}$$

where x_i^{k+1} and x_i^k denote the forward and backward locations, at t^{k+1} and t^k, respectively, of a particle moving along its pathline.

Furthermore, we approximate the time derivative of the velocity by a first order backward difference

$$\frac{Dv_i}{Dt} \cong \frac{v_i^{k+1} - {}^k v_i}{\Delta t}, \tag{53}$$

where ${}^k v_i$ denotes the known velocity at time t^k at an as yet unknown x_i^k, so that at the end of the time step it will reach the known x_i^{k+1} location.

In view of (53), we now approximate (49) and obtain

$$v_i^{k+1} = {}^k v_i - g\,\Delta t \frac{\partial \Phi}{\partial x_j} T_{ij}. \tag{54}$$

Substituting (54) into (46), yields a mass balance equation in the form

$$U_\beta \frac{\partial \Phi}{\partial t} = \left(\frac{\Delta t}{\rho\beta}\right) \frac{\partial^2 \Phi}{\partial x_i\,\partial x_j} T_{ij} - \left({}^k v_i + g\,\Delta t \frac{\partial Z}{\partial x_j} T_{ij}\right) \frac{\partial \Phi}{\partial x_i} + Q. \tag{55}$$

Equation (55) has the form of a transport equation with a source term Q in the form

$$Q = g\,\Delta t \frac{\partial \Phi}{\partial x_i} \frac{\partial \Phi}{\partial x_j} T_{ij} + {}^k v_i \frac{\partial Z}{\partial x_i}. \tag{56}$$

Note that for small values of the potential gradient $(\partial\Phi/\partial x_j)$, the source term in (56) may be approximated by

$$Q \cong {}^k v_i \frac{\partial Z}{\partial x_i}, \tag{57}$$

in which case (55) will be linear.

It may be advantageous to convert the Eulerian representation of (55) into a

Langrangian one. To do so, we rewrite (55) in the form

$$U_\beta\left[\frac{\partial\Phi}{\partial t}+U_\beta^{-1}\left({}^k\nu_i+g\,\Delta t\frac{\partial Z}{\partial x_j}\,T_{ij}\right)\frac{\partial\Phi}{\partial x_i}\right]=\left(\frac{\Delta t}{\rho\beta}\right)\frac{\partial^2\Phi}{\partial x_j\,\partial x_i}+Q. \tag{58}$$

We define a time derivative operator of the form

$$\frac{\mathscr{L}}{\mathscr{L}t}\equiv\frac{\partial}{\partial t}+\tilde{\nu}_i\frac{\partial}{\partial x_i} \tag{59}$$

so that in view of (59), we rewrite (58) in the form

$$U_\beta\frac{\mathscr{L}\Phi}{\mathscr{L}t}=\left(\frac{\Delta t}{\rho\beta}\right)\frac{\partial^2\Phi}{\partial x_j\,\partial x_i}+Q \tag{60}$$

$\tilde{\nu}_l$ denotes the velocity along a characteristic pathline given by

$$\tilde{\nu}_i=U_\beta^{-1}\left({}^k\nu_i+g\,\Delta t\frac{\partial Z}{\partial x_i}\,T_{ij}\right). \tag{61}$$

The characteristic pathline equation, is given by

$$\frac{\mathscr{L}x_i}{\mathscr{L}t}=\tilde{\nu}_i. \tag{62}$$

The potential Φ can now be solved along its pathline (62) using its governing equation (60) during the time period given by (42), thus yielding the propagation of pressure in the course of time.

3.3. PERIOD DOMINATED BY INERTIAL AND MOMENTUM DIFFUSIVE FLUX

Actually we are now referring to the case when all A_m scalar factors ($m=1,2,\ldots,7$) in (28), are of the same order of magnitude. Hence, the Eulerian presentation of the momentum and mass balances are described by (6) and (13), respectively. In view of Hubberts potential derivatives, (44), (45), and (48), as a material derivative, we rewrite (6) in Lagrangian form

$$n\rho\frac{D\nu_i}{Dt}+n\rho g\frac{\partial\tilde{\varphi}}{\partial x_j}\,T_{ij}-\mu\left(\frac{\partial^2 q_{ri}}{\partial x_j\,\partial x_j}+\frac{\partial^2 q_{rj}}{\partial x_i\,\partial x_j}\right)-\lambda''\frac{\partial^2 q_{ri}}{\partial x_j\,\partial x_j}+$$
$$+\frac{\mu C_f}{\Delta^2}\,\alpha'_{ij}q_{rj}+n\rho g\frac{\partial Z}{\partial x_j}\,T_{ij}=0, \tag{63}$$

where $\tilde{\Phi}$ denotes a reference potential, defined by

$$\tilde{\Phi}=\Phi-Z=\int_{P_0}^{P}\frac{\mathrm{d}p}{g\rho(p)}. \tag{64}$$

In view of (46) and (64) we rewrite the mass balance equation in its Lagrangian form

$$\tilde{S}_0 \frac{\tilde{\mathscr{L}}\tilde{\Phi}}{\tilde{\mathscr{L}}t} + \frac{\partial q_i}{\partial x_j} = 0, \tag{65}$$

where

$$\frac{\tilde{\mathscr{L}}}{\tilde{\mathscr{L}}t} \equiv \frac{\partial}{\partial t} + (U_\beta^{-1} v_j) \frac{\partial}{\partial x_j}, \tag{66}$$

and

$$q_j = n v_j; \qquad \tilde{S}_0 = \rho g S_0 \tag{67}$$

Equations (63) and (65) are now solved simultaneously for the solution of the fluid's velocity v_i and its reference potential $\tilde{\Phi}$.

3.4. PERIOD DOMINATED BY VISCOUS CREEP FLOW

In this case, the drag associated with viscosity dominates the flow regime, after all inertial effects have subsided. In our formulation, this occurs when

$$A_3 > O(1), \qquad A_5 > O(1), \qquad A_6 > O(1), \qquad A_7 > O(1), \tag{68}$$

Hence, while the mass balance equation remains (65) the momentum balance one, in view of (68) and (63) becomes

$$n\rho g \frac{\partial \Phi}{\partial x_j} T_{ij} - \mu \left[\frac{\partial^2 q_{ri}}{\partial x_j \, \partial x_j} \left(1 + \frac{\lambda''}{\mu}\right) + \frac{\partial^2 q_{rj}}{\partial x_i \, \partial x_j} \right] + \frac{\mu C_f}{\Delta_f^2} \alpha'_{ij} q_{rj} = 0. \tag{69}$$

Equation (69) is similar to the Brinkman equation often used by chemical engineers. In (69) the second term expresses internal friction in the fluid while the third one represents friction at the solid-fluid interface. When, in view of (21) and (22.1),

$$\frac{1}{C_f} \left(\frac{\Delta_f}{L_c^v} \right)^2 \gg 1, \tag{70}$$

the second term in (69) can be neglected and (69) converts to a linear momentum balance equation, known as Darcy's law

$$n\rho g \frac{\partial \Phi}{\partial x_j} T_{ij} + \frac{\mu C_f}{\Delta_f^2} \alpha'_{ij} q_{rj} = 0. \tag{71}$$

In view of (71), we may express the flux, q_{rj}, in terms of the potential gradient

$$q_{rj} = -K_{ij} \frac{\partial \Phi}{\partial x_j}, \tag{72}$$

where K_{ij} denotes the hydraulic conductivity tensor defined by

$$K_{ij} = \frac{ng\,\Delta_f^2}{v C_f} (\alpha'_{ij})^{-1} T_{ij}. \tag{73}$$

The relative flux, q_{rj}, is related to fluid's flux, q_j, and solid's velocity, ν_s, by

$$q_{rj} = q_j - n\nu_{sj}. \tag{74}$$

Hence, in view of (72), (74) and the mass balance (65), we obtain

$$\tilde{S}_0 \frac{\tilde{\mathscr{L}}\bar{\tilde{\Phi}}}{\tilde{\mathscr{L}}t} = \frac{\partial}{\partial x_i}\left[K_{ij}\frac{\partial}{\partial x_j}(\bar{\tilde{\Phi}} + Z) - n\nu_{si}\right]. \tag{75}$$

A further simplification of (75) is obtained when the solid's velocity is much smaller than the fluid's one, i.e. when

$$\nu_j \gg \nu_{sj} \tag{76}$$

We may then rewrite (75) in the form

$$\tilde{S}_0 \frac{\tilde{\mathscr{L}}\bar{\tilde{\Phi}}}{\tilde{\mathscr{L}}t} = \frac{\partial}{\partial x_i}\left[K_{ij}\frac{\partial}{\partial x_j}(\bar{\tilde{\Phi}} + Z)\right]. \tag{77}$$

Equation (77) has to be solved for $\bar{\tilde{\Phi}}$, subject to its appropriate initial and boundary conditions.

4. Conclusions

The fluid's mass and momentum balance equations in a saturated porous medium were averaged over time and space. A nondimensionalizing procedure was then applied, accounting also for characteristic distances and time steps associated with extensive variables, their gradients and time derivatives.

The evolution of dominating mass and momentum balance in response to a blunt pressure impact was then analyzed. The governing nondimensional terms in the mass and momentum balances were found to be scaled by two new scalar factors. The first, U_p, gives the volume of fluid added to storage per unit volume of fluid as pressure increases. The second, U_β, represents the part of the change in fluid storage due to the compressibility of the fluid, alone.

As time proceeds, after applying an abrupt pressure change, it was shown that:

- At the very first time instant the compressible fluid acts as an incompressible one resulting in a uniform pressure distribution,
- during a second time interval, the inertial terms dominate and the governing momentum balance equation takes on a wave form,
- during a third time period, the viscous effects (i.e., drag force) arise and the momentum balance is expressed by the entire Navier–Stokes equation,
- as time proceeds, during the fourth stage, inertial terms subside and the viscous ones dominate. The momentum balance equation reduces to a form, similar to Brinkman's equation. However, when the fluid's internal friction is much smaller than that taking place at a solid–fluid interface, a linear

relation between potential gradient and flux is obtained. This linear momentum balance equation is known as Darcy's equation.

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